Graded rings and varieties in weighted projective space

Miles Reid

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Abstract

This chapter is a first introduction to weighted projective spaces (wps) $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ and the Proj correspondence

$$\begin{matrix} \text{projective variety} \\ X \subset \mathbb{P} \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} \text{graded ring} \\ R = k[x_0, \dots, x_n]/I \end{matrix} \tag{1}$$

The correspondence (1) between geometry and algebra is a minor but very fruitful generalisation of the usual idea of varieties in *straight* projective space $\mathbb{P}^n = \mathbb{P}(1, \dots, 1)$. The simple device of working with varieties contained in the ready-made ambient spaces $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ allows us in many cases to by-pass the definition of abstract variety (or more general schemes) at the cost of a bit of messing around with weighted homogeneous polynomials, so that projective varieties in wps are technically not really much harder than affine varieties. Practically every item in this chapter relates in a transparent way to something in the treatment of subvarieties of straight projective space; compare [UAG], Chapter 5 and Hartshorne [H], Chapter I. Nontrivial weights $a_i > 1$ leads naturally to cyclic quotient singularities, $\mathbb{Q}$-divisors and cyclic orbifold behaviour.

Weighted projective spaces have appeared implicitly in algebraic geometry since ancient times; the most basic example is a hyperelliptic curve $y^2 = f_{2g+2}(x)$ viewed as a double cover of $\mathbb{P}^1$, that is, a weighted hypersurface $C_{2g+2} \subset \mathbb{P}(1, 1, g+1)$. The general definition was codified in Grothendieck's notion of $\operatorname{Proj} R$ ([EGA2], see also [H], Chapter II, Section 7), and is a basic ingredient of modern work on algebraic surfaces and 3-folds.

1 Weighted projective space $\mathbb{P}(a_0, \dots, a_n)$

The definition is similar to straight projective space: wps is the quotient

$$\mathbb{P}(a_0, \dots, a_n) = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m$$

of $\mathbb{A}^{n+1}$ under the equivalence relation

$$(x_0, \dots, x_n) \sim (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n) \quad \text{for } \lambda \in \mathbb{G}_m. \tag{2}$$

(Here $\mathbb{G}_m$ is the multiplicative algebraic group, that is, the variety whose k-points are the multiplicative group $k^\times$.) We can usually assume that the a_i are mostly coprime (see 3.1 below); a typical condition is that no n of the a_i have a common factor.

Example 1.1 $\mathbb{P}(1,1,a)$ is the cone over the rational normal curve of degree a in $\mathbb{P}^a$. We have already met this surface in [Ch], 2.9 as the image of the surface scroll $\mathbb{F}(a,0)$, under which the negative section is contracted. Write x_1, x_2, y for coordinates on $\mathbb{A}^3$. The quotient of $\mathbb{A}^3 \setminus 0$ by $\mathbb{G}_m$ is realised by the morphism $(\mathbb{A}^3 \setminus 0) \to \mathbb{P}^{a+1}$ defined by

$$(x_1, x_2, y) \mapsto (x_1^a : x_1^{a-1} x_2 : \cdots : x_2^a : y).$$

The cone point is $(0,0,1)$, the equivalence class of the y-axis.

At a point of $\mathbb{A}^3$ with $x_1 \neq 0$, setting $x_1 = 1$ defines a *slice* of the action of $\mathbb{G}_m$ (compare [UAG], Figure 1.4 – here slice means a local submanifold that provides a unique choice of representative in each equivalence class), so that the quotient $(\mathbb{A}^3 \setminus (x_1 = 0))/\mathbb{G}_m$ is just $\mathbb{A}^2$ with coordinates $\frac{x_2}{x_1}, \frac{y}{x_1^a}$. Similarly for $x_2 \neq 0$. However, near the y-axis, the group action does not have a slice: indeed, it is given by

$$(x_1, x_2, y) \mapsto (\lambda x_1, \lambda x_2, \lambda^a y),$$

so that a point of the y-axis is fixed by roots of unity $\mu_a \subset \mathbb{G}_m$ (a cyclic group of order a). Setting $y = 1$ cuts the y-axis in one point, but cuts neighbouring orbits in a points $(\varepsilon x_1, \varepsilon x_2, y)$ for $\varepsilon \in \mu_a$. Thus as coordinates on the quotient $(\mathbb{A}^3 \setminus (y = 0))/\mathbb{G}_m$, I must take

$$\frac{x_1^a}{y}, \frac{x_1^{a-1} x_2}{y}, \dots, \frac{x_2^a}{y}.$$

This is a model for the standard cyclic quotient behaviour in 2.2.

Example 1.2 Consider the equation $y^2 = f_{2a}(x_1, x_2)$, where f is a homogeneous polynomial having $2a$ distinct roots. This defines the hypersurface $C_{2a} \subset \mathbb{P}(1,1,a)$ which is the general hyperelliptic curve of genus $g = a - 1$. Because of the monomial y^2, the hypersurface does not pass through the cone point $(0,0,1)$. The curve C is the union of two affine pieces given by

$x_1 = 1$ and $x_2 = 1$, glued together in the obvious way, so that $C \to \mathbb{P}^1$ is the double cover with $2a$ branch points $f = 0$. Note that it is not a wise move to take the projective closure of $y^2 = f_{2a}(x)$ in straight $\mathbb{P}^2$ – it leads to a complicated singularity at infinity, and general confusion.

One can view the equation $y^2 = f_{2a}(x_1, x_2)$ as a general quadric section of $\overline{\mathbb{F}} = \mathbb{P}(1,1,a) \subset \mathbb{P}^a$, or a general curve $C \in |2aA+2B|$ in $\mathbb{F}_a$ in the set-up of [Ch], 2.9. One calculates the canonical class $K_C = (a-2)A$ and the genus $g = a-1$ in any number of ways (compare Example 4.5).

2 Graded rings and $\operatorname{Proj} R$

Definition 2.1 A *graded ring* $R = \bigoplus_{n\geq 0} R_n$ is a ring R whose multiplication $R \times R \to R$ respects the grading, taking $R_n \times R_m \to R_{n+m}$. It is sometimes useful to work with a grading taking values in more general semigroups, but here I restrict attention to gradings by $n \in \mathbb{Z}$ with $n \geq 0$. In view of the intended applications to varieties over a field k, I impose the following additional conditions:

(i) $R_0 = k$ is the ground field;

(ii) R is finitely generated as a ring over k;

(iii) R is an integral domain.

More general cases may be interesting for several different purposes.

Since every element of R is a sum of homogeneous pieces, it follows from (ii) that the generators of R can be chosen to be finitely many homogeneous elements x_i of degree $a_i > 0$. The key example is the polynomial ring $k[x_0, \dots, x_n]$ where $\operatorname{wt} x_i = a_i$. Every polynomial is a sum of monomials $x^{\mathbf{m}} = \prod x_i^{m_i}$, having weight $\sum m_i a_i$. A polynomial f is *homogeneous* (also *weighted homogeneous* or *quasihomogeneous*) of weight d if every monomial in f has weight d. An ideal in a graded ring $I \subset R$ is *graded* or *weighted homogeneous* if I is the sum of its homogeneous components, $I = \bigoplus_{n\geq 0} I_n$ with $I_n = I \cap R_n$. It is equivalent to say that I is generated by (finitely many) homogeneous elements. Thus in general, the rings R considered have the form $R = k[x_0, \dots, x_n]/I$, where $\deg x_i = a_i$ and I is a homogeneous prime ideal.

2.1 Construction of $\operatorname{Proj} R$

A ring $R = k[x_0, \dots, x_n]/I$ corresponds to an irreducible affine variety $\mathcal{C}X = \operatorname{Spec} R = V_a(I) \subset \mathbb{A}^{n+1}$; the subscript a stands for affine or inhomogeneous. $\mathcal{C}X$ is the (weighted homogeneous) *affine cone* over the projective variety $V_h(I) = X$ defined below. A polynomial $f(x_0, \ldots, x_n)$ is weighted homogeneous of degree d if and only if

$$f(\lambda^{a_0}x_0, \lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n) = \lambda^d f(x_0, \ldots, x_n) \quad \text{for all } \lambda \in \mathbb{G}_m,$$

so that the condition $f(P) = 0$ is well defined on equivalence classes of $\sim$ in (2). One can thus define the projective variety or *homogeneous spectrum* $X = \operatorname{Proj} R = V_h(I) \subset \mathbb{P}(a_0, \ldots, a_n)$ as the quotient $(V_a(I) \setminus 0)/\mathbb{G}_m$.

I want to construct the quotient X as an algebraic variety. So what are the *functions* on X? In the elementary spirit of [UAG], Chapter 5, one can approach this via the rational function field consisting of ratios of homogeneous elements of the same degree d

$$k(X) = \left\{ \frac{g}{h} \;\middle|\; g, h \in R_d \right\}/\sim \quad \text{where} \quad \frac{g}{h} \sim \frac{g'}{h'} \iff gh' - hg' \in I,$$

and define a rational function $f \in k(X)$ to be *regular* at $P \in X$ if there exists an expression $f = g/h$ with $h(P) \neq 0$.

As an alternative, for any $d > 0$ and any homogeneous element $f \in R_d$, define the *principal open set* $X_f \subset X$ by $X_f := \left\{ P \in X \;\middle|\; f(P) \neq 0 \right\}$. Then X_f is an affine variety having coordinate ring

$$k[X_f] = \left(R\left[\frac{1}{f}\right] \right)^0 = \left\{ \frac{g_{md}}{f^m} \;\middle|\; g \in R_{md} \right\}. \tag{3}$$

The subscript ^{0} means homogeneous of degree 0, that is, $\mathbb{G}_m$-invariant.

In other words, what is going on here is a systematic construction of the quotient $(\mathcal{C}X \setminus 0)/\mathbb{G}_m$: the open sets $(f \neq 0) \subset \mathcal{C}X$ for $f \in R_d$ provide arbitrarily small $\mathbb{G}_m$-invariant affine open sets covering $\mathcal{C}X \setminus 0$. For every such open set, take the set theoretic quotient, and make it an affine quotient variety by taking the ring (3) of all $\mathbb{G}_m$-invariant fractions as its coordinate ring.

2.2 Local affine coordinates

Since every point of $\mathcal{C}X \setminus 0$ has at least one $x_i \neq 0$, the quotient X is more modestly covered by the standard affine pieces $X^{(i)} = (x_i \neq 0)$. I treat first the construction of $\mathbb{P}(a_0, \ldots, a_n)$ for simplicity, so take $R = k[x_0, \ldots, x_n]$. Then the affine ring (3) is conveniently described as a ring of invariants for the cyclic group $\mathbb{Z}/a_i$ acting on a polynomial ring.

The basic idea, just as in the homogeneous–inhomogeneous trick for straight projective space, is that I want to set $x_i = 1$ on the affine piece $x_i \neq 0$. However, before doing that, I first adjoin the a_ith root of x_i, setting $x_i = \xi_i^{a_i}$, so that $\operatorname{wt}\xi_i = 1$; the point of doing this is to be able replace each x_j by a homogeneous ratio of degree 0. For clarity, suppose $i = 0$. Since $\xi_0 = \sqrt[a_0]{x_0}$ has weight 1, each x_i now occurs in a homogeneous ratio of degree 0 with only ξ_0 in the denominator, namely $x_i/\xi_0^{a_i}$. Thus setting $\xi_0 = 1$ amounts to replacing each x_j by this ratio. After adjoining ξ_0, the ring (3) of homogeneous rational forms of degree 0 is the polynomial ring

$$k[x_1^{(0)}, \dots, x_n^{(0)}], \quad \text{where} \quad x_i^{(0)} = \frac{x_i}{\xi_0^{a_i}} \quad \text{for } i = 1, \dots, n. \tag{4}$$

To get what I want, I still need to get rid of the irrational quantity ξ_0. For this, note that adjoining $\xi_0 = \sqrt[a_0]{x_0}$ is a cyclic Galois extension of rings, with Galois group $\xi_0 \mapsto \varepsilon\xi_0$, where $\varepsilon \in \mu_{a_0}$; here $\mu_{a_0} \subset \mathbb{G}_m$ is the group of a_0th roots of 1, which is a cyclic group of order a_0 (assuming characteristic coprime to a_0). We can eliminate the irrationality ξ_0 by passing to the ring of invariants of μ/a_0 acting by $x_i^{(0)} \mapsto \varepsilon^{-a_i} x_i^{(0)}$. The conclusion is that the affine piece $x_0 \neq 0$ of $\mathbb{P}(a_0, \dots, a_n)$ is the quotient of $\mathbb{A}^n$ by the action $x_i^{(0)} \mapsto \varepsilon^{-a_i} x_i^{(0)}$, that is, the quotient $\frac{1}{a_0}(a_1, \dots, a_n)$.

Remark 2.2 The discussion here was at the algebraic level, concerned with difficulty of writing down all the homogeneous ratios involving a variable x_0 of degree $a_0 > 0$. The point, however, is exactly the same as the geometric difficulty of 1.1 of not being able to find a slice of the group action (at a point of the x_0-axis whose stabiliser group jumps up compared to its neighbours). Introducing $\sqrt[a_0]{x_0}$ also provides a finite cyclic covering space on which the $\mathbb{G}_m$ action extends to an action having a slice.

3 Truncated rings $R^{[d]}$ and Veronese embedding

The dth truncated ring is the subring $R^{[d]} \subset R$ defined by

$$(R^{[d]}) = \bigoplus_{d|n} R_n = \bigoplus_{i \geq 0} R_{di}.$$

In other words, we take only the homogeneous pieces of R of degree divisible by d. Although we've passed to a smaller ring, $\operatorname{Proj} R$ does not change up to isomorphism, because any homogeneous ratio in R can be expressed as a homogeneous ratio in $R^{[d]}$.

There are two different conventions in use about degrees in $R^{[d]}$: we can view the elements of R_{di} as having the same degree di in the truncated ring as they had in R, or we can divide degrees through by d. It is common for either convention to be in force in papers, sometimes both within the same argument.

Example 3.1 $R = k[x_0, x_1]$ has $\operatorname{Proj} R = \mathbb{P}^1$ (with coordinates x_0, x_1). The truncated ring $R^{[2]} = k[x_0^2, x_0x_1, x_1^2]$ is the homogeneous coordinate ring of the plane conic $C := (u_0u_2 = u_1^2) \subset \mathbb{P}^2$. This is a very familiar argument: at every point of C either $u_0 \neq 0$, and then the local parameter is $x_1/x_0 = u_1/u_0$, or $u_2 \neq 0$, and then $x_0/x_1 = u_1/u_2$ is a local parameter.

The same applies to all the Veronese embeddings

$$v_d \subset \mathbb{P}^n \hookrightarrow \mathbb{P}^N \quad \text{where } N + 1 = \binom{n+d}{n};$$

most famously, the surface $v_2 \colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ embedded by

$$(x_0, x_1, x_2) \mapsto (x_0^2, x_0x_1, \dots, x_2^2).$$

Example 3.2 In Example 1.1, we embedded $\mathbb{P}(1, 1, a) \hookrightarrow \mathbb{P}^{a+1}$ by taking

$$(x_1, x_2, y) \mapsto (x_1^a, x_1^{a-1}x_2, \dots, x_2^a, y).$$

Both $\mathbb{P}(1, 1, a)$ and its image $\overline{\mathbb{F}} \subset \mathbb{P}^{a+1}$ have advantages: $\mathbb{P}(1, 1, a)$ corresponds to the polynomial ring $k[x_1, x_1, y]$, with only 3 generators, albeit of different weights, and no relations between the generators. The image $\overline{\mathbb{F}}$ is a straight projective space, with $a + 1$ homogeneous coordinates (of the same degree 1 or a as you like), and an obvious set of defining relations. $\overline{\mathbb{F}}$ is a cone with a ruling $|L|$ by generating lines, and a rational projection to the base of the cone $\cong \mathbb{P}^1$. You can recover the structure of $\mathbb{P}(1, 1, a)$ by taking the linear system or generating lines, and noting that the hyperplane section of $\overline{\mathbb{F}} \subset \mathbb{P}^{a+1}$ is linearly equivalent to aL.

Proposition 3.3 (Veronese embedding) *For a graded ring as in Definition 2.1,*

$$\operatorname{Proj} R^{[d]} \cong \operatorname{Proj} R \quad \textit{for any } d > 0.$$

Proof Any homogeneous element $f \in R_m$ has a power $f^d \in R^{[d]}$, and the ring of fractions $R^{[d]}[1/f^d]$ is isomorphic to $R[1/f]$; this is obvious because

$$\frac{g_{mi}}{f^i} = \frac{f^{dj-i} g_{mi}}{f^{dj}}$$

(where $dj \geq i$). In other words, any homogeneous ratio of elements of R can be written as a homogeneous ratio of elements of the truncated ring $R^{[d]}$. On the other hand, $\operatorname{Proj} R$ and $\operatorname{Proj} R^{[d]}$ are both constructed just from these rings of fractions by taking elements of degree 0 then Spec. QED

3.1 Applications

Proposition 3.3 has two applications: to reduce to the "straight" case when R generated in degree 1, and to reduce to the "well formed" case when there is no orbifold behaviour in codimension 1.

Proposition 3.4 *For a graded ring as in Definition 2.1, there exists a truncation $R^{[d]}$ which is generated by its elements of the smallest degree; in this case, one would normally divide degrees by d, and say that $R^{[d]}$ is generated by elements of* deg 1 *(except in cases where some confusion may arise).*

The point is purely combinatorial: given $\operatorname{wt} x_i = a_i$, there exists some d so that every monomial of degree md can be written as a product of elements of degree d. The idea of the proof is to set $X_i = x_i^{N/a_i}$, where $N = \operatorname{lcm}(a_i)$, and argue on R as a module over the graded ring $k[X_1, \dots, X_n]$, proving various finiteness assertions for it. I omit the proof in this draft (see [EGA2], Lemma 2.1.6).

Definition 3.5 (well formed wps) A wps $\mathbb{P}(a_0, \dots, a_n)$ is *well formed* if no $n-1$ of $a_0, a_1, \dots, a_n$ have a common factor.

Proposition 3.6 *Consider the weighted polynomial ring $R = k[x_0, \dots, x_n]$, where $a_0, \dots, a_n$ are positive integers with* $\operatorname{wt} x_i = a_i$. *Then*

(I) If d is a common factor of all a_i then $R^{[d]} = R$; thus $\mathbb{P}(a_0, \dots, a_n) = \mathbb{P}(a_0/d, \dots, a_n/d)$.

(II) Suppose that $a_0, \dots, a_n$ have no common factor, and that d is a common factor of all a_i for $i \neq j$ (and therefore coprime to a_j). Then the dth truncation of R is the polynomial ring

$$R^{[d]} = k[x_0, \dots, x_{j-1}, x_j^d, x_{j+1}, \dots, x_n].$$

Thus, in this case

$$\mathbb{P}(a_0, \dots, a_n) = \mathbb{P}\Big(\frac{a_0}{d}, \dots, \frac{a_{j-1}}{d}, a_j, \frac{a_{j+1}}{d}, \dots, \frac{a_n}{d}\Big).$$

In particular, by passing to a truncation $R^{[d]}$ of R which is a polynomial ring generated by pure powers of x_i, we can always write any wps as well formed.

Proof If $d \mid a_i$ for every i then the degree of every monomial is divisible by d, so that (I) is obvious. In this case, truncation does not change anything.

If $d \mid a_i$ for every $i \neq j$ then $x_i \in R^{[d]}$ for every $i \neq j$, but the only way that x_j can occur in a monomial of degree divisible by d is as a dth power. Thus $R^{[d]}$ is as in (II). Replacing

$$R = k[x_0, \dots, x_j, \dots, x_n] \quad \text{by} \quad R^{[d]} = k[x_0, \dots, x_j^d, \dots, x_n]$$

changes the ring R, but does not change $\operatorname{Proj} R$. The point is that since $d \mid a_i$ for $i \neq j$ and is coprime to a_j, the only way that x_j can appear in a homogeneous ratio with other x_i is as an expression in x_j^d. QED

Well formed is equivalent to the condition that the quotient morphism $\mathbb{A}^{n+1} \to \mathbb{P}(a_0, \dots, a_n)$ does not have orbifold behaviour along any coordinate hyperplane $H : (x_i = 0)$.

Example 3.7 Consider the weighted projective plane $\mathbb{P}(bc, ac, ab)$ with coordinates x, y, z, where a, b, c are coprime integers; then $\mathbb{P}(bc, ac, ab) \to \mathbb{P}^2$ defined by $(x, y, z) \mapsto (x^a, y^b, z^c)$ is an isomorphism. In other words, although the rings $k[x^a, y^b, z^c] \subset k[x, y, z]$ are of course not equal, because of the way the weights are arranged, the two rings provide exactly the same opportunities for forming weighted homogeneous ratios.

Now consider the quotient ring $R = k[x, y, z]/(x^a + y^b + z^c)$; then $\operatorname{Spec} R$ is the singularity $X : (x^a + y^b + z^c = 0) \subset \mathbb{A}^3$. However, $\operatorname{Proj} R \cong \operatorname{Proj} R^{[abc]}$ is the line $\mathbb{P}^1 \subset \mathbb{P}^2$. In this case the $\mathbb{G}_m$ action on $\mathbb{A}^3$ has the nontrivial stabiliser subgroups μ_a at every point of the coordinate line $x = 0$, etc., and the quotient morphism $(X \setminus 0) \to \mathbb{P}^1 \subset \mathbb{P}^2$ has orbifold points of order a at the intersection of $\mathbb{P}^1$ with the coordinate lines $x = 0$, etc.

A famous case is the E_8 singularity $X : (x^2 + y^3 + z^5 = 0)$, which is naturally weighted homogeneous with weights $15, 10, 6$. The $\mathbb{G}_m$ quotient morphism $X \to \mathbb{P}^1$ defined by the ratio $x^2 : y^3 : z^5$ has stabiliser of order 2, 3 and 5. The weighted blowup $Y \to X$ (the graph of the quotient morphism $X \to \mathbb{P}^1$) is a surface having cyclic quotient singularities of order $2, 3, 5$ at the 3 points, giving rise to the Dynkin diagram of E_8.

Remark 3.8 In fact in this case, we can recover the ring R from its Proj together with its orbifold structure (this probably doesn't make sense at present, but will be explained in a later chapter): let $C = \mathbb{P}^1 \subset \mathbb{P}^2$, marked with three points P_x, P_y, P_z of order $2, 3, 5$. The orbifold canonical class

$$K_C + \frac{1}{2}P_x + \frac{2}{3}P_y + \frac{4}{5}P_z$$

has degree $\frac{1}{30}$, and one can check $R(C, K_C + \frac{1}{2}P_x + \frac{2}{3}P_y + \frac{4}{5}P_z)$ has generators x, y, z in degree $15, 10, 6$.

4 Hilbert series and applications

Definition 4.1 (Hilbert function and Hilbert series) Given a graded ring R, the *Hilbert function* is the numerical function

$$\mathbb{Z} \to \mathbb{Z} \quad \text{given by} \quad d \mapsto P_d(R), \quad \text{where } P_d(R) = \dim_k R_d.$$

The *Hilbert series* of R is the formal power series $P_R(t) = \sum_{d \geq 0} P_d t^d$.

It usually happens that P_R is a rational function with denominator $\prod_{i=0}^n (1 - t^{a_i})$ where R has generators of degree a_i.

Example 4.2 The straight polynomial ring $k[x_0, \dots, x_n]$ has Hilbert function $P_d = \binom{n+d}{n}$. The Hilbert series is

$$P(t) = \sum_{d \geq 0} \binom{n+d}{n} t^d = \frac{1}{(1-t)^{n+1}}.$$

The power series expansion is well known, but it can be calculated as follows (say, when $n = 2$):

$$P(t) = \sum P_d t^d = 1 + 3t + 6t^2 + 10t^3 + \cdots + \binom{d+2}{2} t^n + \cdots$$

So by long multiplication

$$\begin{aligned} (1-t)P(t) = \sum P_d t^d &= 1 + 3t + 6t^2 + 10t^3 + \cdots + \binom{d+2}{2} t^d + \cdots \\ &\quad - t - 3t^2 - 6t^3 - \cdots - \binom{d+1}{2} t^d - \cdots \\ &= 1 + 2t + 3t^2 + 4t^3 + \cdots + (d+1)t^d + \cdots \end{aligned}$$

Repeating another couple of times gives $(1-t)^3 P(t) = 1$ as required.

Proposition 4.3 *The weighted polynomial ring* $k[x_0,\dots,x_n]$ *with weights* $a_0,\dots,a_n$ *has Hilbert series*

$$P(t)=\frac{1}{\prod_{i=0}^n(1-t_i^{a_i})}\,.$$

Proof For a single variable,

$$\frac{1}{(1-x_i)}=1+x_i+x_i^2+\cdots\,.$$

The rhs is just a list of every monomial in $k[x_i]$ counted once each. Taking the product of these expressions over each i gives

$$\frac{1}{\prod(1-x_i)}=\prod\frac{1}{(1-x_i)}=\sum x^{\mathbf{m}},$$

where the sum on the rhs consists of every monomial $x^{\mathbf{m}}=x_0^{m_0}x_1^{m_1}\cdots x_n^{m_n}$ in $k[x_0,\dots,x_n]$ counted once each. If we substitute $x_i=t^{a_i}$ in this formal expression, each monomial $x^{\mathbf{m}}$ contributes one summand $t^{\operatorname{wt}x^{\mathbf{m}}}$ to the rhs. QED

The proposition says that the number P_k of monomomials of weight k in $k[x_0,\dots,x_n]$ equals the coefficient of t^k in the stated power series $P(t)$. Calculating the terms of the power series is exactly the same problem as calculating the set of monomials of weight k, so this is a convenient way of holding the information of the numerical function P_k, but does not itself make the calculation any easier.

Example 4.4 The hypersurface ring $R=k[x_0,\dots,x_n]/(f_d)$ has Hilbert series

$$P(t)=(1-t^d)/\prod(1-t^{a_i})$$

and a weighted c.i. of degree $d_1,\dots,d_k$ gives

$$P(t)=\prod_{j=1}^k(1-t^{d_j})/\prod_{i=0}^n(1-t^{a_i})$$

Example 4.5 Let C be a hyperelliptic curve of genus g with the linear system $|A|=g_2^1$. By Clifford's theorem, this is the most special of all special linear systems, and the dimension of $H^0(C, kA)$ is completely determined by RR[1]:

$$h^0(C, kA) = \begin{cases} k+1 & \text{for } k \le g, \\ 1-g+2k & \text{for } k \ge g. \end{cases}$$

Thus the graded ring $R = \bigoplus H^0(C, kA)$ has Hilbert series

$$P(t) = 1 + 2t + 3t^2 + \cdots + gt^{g-1} + (g+1)t^g + (g+3)t^{g+1} + \cdots$$

Doing long multiplication by $1-t$ a couple of time as in Example 4.2 gives

$$\begin{aligned} (1-t)P(t) &= 1 + t + t^2 + \cdots + t^g + 2t^{g+1} + \cdots \\ (1-t)^2P(t) &= 1 + t^{g+1}. \end{aligned}$$

Thus

$$P(t) = \frac{1+t^{g+1}}{(1-t)^2} = \frac{1-t^{2g+2}}{(1-t)^2(1-t^{g+1})}.$$

This is the Hilbert series of the weighted hypersurface $C_{2a} \subset \mathbb{P}(1,1,a)$ where $a = g+1$.

5 More important examples

5.1 $\mathbb{P}(1,2,3)$

This can be treated in several ways: write $k[x,y,z]$ for the polynomial ring with $\operatorname{wt} x = 1, \operatorname{wt} y = 2, \operatorname{wt} z = 3$, so $\mathbb{P}(1,2,3) = \operatorname{Proj} k[x,y,z]$.

1. The general definition of $\mathbb{P}(a_0,\dots,a_n)$ as a $\mathbb{G}_m$ quotient and the local coordinate trick 2.2 shows that $\mathbb{P}(1,2,3)$ is covered by 3 affine pieces

$\mathbb{A}^2$	with coordinates	$y/x^2, z/x^3$
$\frac{1}{2}(1,1)$	"	$x/\eta, z/\eta^3$
$\frac{1}{3}(1,2)$	"	$x/\zeta, y/\zeta^2$

[1]For a divisor D on a nonsingular curve C of genus g, RR says

$$h^0(C,D) - h^0(C, K_C - D) = 1 - g + \deg D.$$

Here $H^0(C,D) = \mathcal{L}(D) = \{f \in k(C) | \operatorname{div} f + D \ge 0\}$ is the Riemann–Roch space of D, with $h^0(C,D) = \dim H^0(C,D)$ and K_C is the canonical divisor. D is a *special divisor* if both $h^0(C,D) \ne 0$ and $h^0(C, K_C - D) \ne 0$. Clifford's theorem says that a special divisor D satisfies

$$\deg D \ge 2(h^0(C,D) - 1),$$

and equality holds (apart from the elementary cases $D = 0$ or $D = K_C$) if and only if C is hyperelliptic and $D = kA$ where $A = g^1_2$.

2. Consider the action of the symmetric group S_3 on ordinary $\mathbb{P}^2$ by permuting the coordinates x_1, x_2, x_3. The quotient morphism is given by $\mathbb{P}^2 \to \mathbb{P}^2/S_3 = \operatorname{Proj} k[x,y,z]$ where

$$(x, y, z) = (x_1 + x_2 + x_3, x_1x_2 + x_2x_3 + x_3x_1, x_1x_2x_3)$$

are the elementary symmetric functions.

3. The order 6 truncation $R^{[6]}$ of $k[x,y,z]$ corresponds to a Veronese embedding of $\mathbb{P}(1,2,3)$ embeds it as a (singular) del Pezzo surface of degree 6 $S_6 \subset \mathbb{P}^6$. Write out the 7 monomials of degree 6 as the Newton polygon

$$\begin{matrix} x^6 & x^4y & x^2y^2 & y^3 \\ x^3z & xyz & & \\ z^2 & & & \end{matrix} \tag{5}$$

It is not hard to write out the 9 quadratic relations between these monomials; they define the image $S_6 \subset \mathbb{P}^6$.

4. Toric variety corresponding to the cone dual to (5).

5.2 Weiestrass model of an elliptic curve

If E is an elliptic curve, and $P \in E$ a marked point (that can serve as the origin of the group law), the graded ring $R(E,P) = \bigoplus_{k\geq 0} H^0(E, kP)$ is of the form $k[x,y,z]/f_6$, and defines an embedding $E \subset \mathbb{P}(1,2,3)$. It is a hyperplane section of the variety in (3) above.

5.3 Double covers

$X_{2a} \subset \mathbb{P}(1,\dots,1,a)$ defined by $y^2 = f_{2a}(x_0,\dots,x_n)$ is a double cover of $\mathbb{P}^n$ branched in the hypersurface $(f_{2a} = 0)$ of degree $2a$.

The hypersurface $X_{2b} \subset \mathbb{P}(1,1,2,b)$ is a double cover of the ordinary quadric cone $\mathbb{P}(1,1,2) = Q \subset \mathbb{P}^3$ ramified in the vertex and in the intersection of Q with a hypersurface of degree $2b$.

6 The hyperplane section theorem

Let R be a graded ring, and $x_0 \in R$ a graded element of degree a_0. Suppose that x_0 is a regular element of R, that is, a non-zerodivisor. Then multiplication by x_0 is an inclusion $R \hookrightarrow R$ with image the principal ideal (x_0), and

I arrive at the exact sequence

$$0 \to (x_0) \to R \to \overline{R} \to 0,$$

where $\overline{R} = R/(x_0)$. Geometrically, if $\operatorname{Proj} \overline{R}$ is the hyperplane section of $\operatorname{Proj} R$ given by $x_0 = 0$.

The hyperplane section principle says that under these assumptions, we can deduce a lot of the structure of R from $\overline{R}$ and vice-versa.

Theorem 6.1 (hyperplane section principle) *1. Let $\overline{x}_1, \dots, \overline{x}_k$ be homogeneous element that generate $\overline{R}$, and $x_1, \dots, x_k \in R$ any homogeneous elements that map to $\overline{x}_1, \dots, \overline{x}_k \in \overline{R}$. Then R is generated by $x_0, x_1, \dots, x_k$.*

2. *Under the assumption of (1), let $\overline{f}_1, \dots, \overline{f}_n$ be homogeneous generators of the ideal of relations holding between $\overline{x}_1, \dots, \overline{x}_k$. Then there exist homogeneous relations $f_1, \dots, f_n$ holding between $x_0, x_1, \dots, x_n$ in R such that the f_i reduces to $\overline{f}_i$ modulo x_0 and $f_1, \dots, f_n$ generate the relation between $x_0, x_1, \dots, x_n$.*

3. *Similar for the syzygies.*

Proof I just give a sketch. For (1), I can choose $x_i \mapsto \overline{x}_i$ by the assumption that $R \to \overline{R}$ is graded and surjective. Given any $y \in R$, suppose that it maps to $\overline{y} \in \overline{R}$. Then $\overline{y} = g(\overline{x}_1, \dots, \overline{x}_k)$ for some homogeneous polynomial g. Taking the same g gives

$$y - g(x_1, \dots, x_k) \in \ker\{R \to \overline{R}\} = (x_0)$$

so that $y - g(x_1, \dots, x_k) = x_0 y'$, where y' has smaller degree than y. Thus by induction, I can assume that y' is in the subring generated by $x_0, \dots, x_k$, and I conclude by induction.

(2) and (3) are similar, and are omitted in this draft.

7 Preview of material of later chapters

The Hilbert syzygies theorem. Cohomology and the Cohen–Macaulay condition. Canonical class, the Gorenstein condition assuming well-formed. Orbifold canonical class and the Gorenstein condition more generally. Unprojection.

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